Implicit Free Surface Algorithms for Global Ocean Modelling

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1 Introduction

The free surface height can be solved from the free surface kinematic boundary condition or from the vertical integrated continuity equation. The former has a complex form because the gravity direction changes from place to place in a Cartesian coordinate system and the property of mass conservation cannot in general be preserved due to discretisation errors. The latter encounters the problem of the vertical integration direction changing across the domain. Here we introduce a new method to solve free surface height in Cartesian co-ordinates. This method has an equation with a relatively simple form and can preserve mass conservation. The free surface kinematic boundary condition will be automatically coupled into the divergence of momentum equation in the finite element discretization. The implicitness of free surface height solver is a very important feature of the current model because the CFL condition associated with free surface waves will in general (in the absence of mode splitting) result in too restrictive time steps especially when unstructured meshes with some small elements are used. We want to use large time step to integrate the control equations with large Courant numbers (> 1). In the following sections, an arbitary coordinate system implicit free surface height algorithm will be described. Thus avoiding (depending on the coordinate system chosen) the pole-singularity problem.

A two-step approach method has been developed here to achieve the accurate balance of geostrophic flow. The free surface height is decomposed into geostrophic part and the rest. The geostrophic part is represented by quadratic basis functions and thus can accurately balance the Coriolois term. Another option is to do implicitness of Coriolis term when discretize the momentum equations. This will help to make the model more stable. The surface and bottom friction has been represented as a implicit absorption term. This improves the model stability especially in shallow water or wetting-drying cases. In addition to this, two extra terms have been devised for wetting-drying case, e.g. an extra pressure term and/or absorption terms are added to the momentum equation to keep a thin layer of water stay in the dried area. These terms tend to zero in deep water so won't affect the consistency of the original set of equations away from wet-drying regions. A sub-grid stabilization method has been devised for the free surface height equations. This method will help to eliminate the spurious oscillations generated by discretization scheme.

The remaining sections of this note are organized as follows. The basic equations of traditional and the new algorithm for free surface are presented in section 2. A two-step approach for the accurate balance of geostrophic flow is described in section 3. The implicit Coriolis term method and implicit absorption term method are described in section 4. Two ways to deal with wetting-drying are presented in section 5. The sub-grid stabilization method is described in section 6.

2 Basic Equations for Solution of Free Surface

For sake of clarity, the continuity and momentum equations are written in the following forms:

$$\nabla \cdot u = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} = R - g\nabla\xi \tag{2}$$

where u is the velocity vector; $R = r + c$. $c = -2\Omega \times u$ is the Coriolis term and r is the lumping of all the rest right hand side terms. Ω is the earth rotation vector. ρ is the reference density under Boussinesq approximation. ξ is free surface height.
Equation(1) and equation(2) can be written in the following semi-discretized form

$$\nabla \cdot u^{n+1} = 0 \tag{3}$$

$$\frac{u^{n+1} - u^n}{\Delta t} = R - g\nabla\xi \tag{4}$$

where the superscripts represent discrete time levels.

2.1 Traditional Wave Equation Used in Ocean Modelling for Calculation of the Free Surface

To get the wave equation (7), we need to integrate continuity equation (1) in vertical direction from bottom to surface

$$\int_b^\xi \nabla \cdot u dr = 0 \tag{5}$$

using free surface and bottom kinematic boundary condition, we get

$$\frac{\partial \xi}{\partial t} + \nabla_h \cdot D\bar{u} = 0 \tag{6}$$

where, $D = \xi - b$ is water depth, over bar means the depth averaging. e.g., $\bar{u} = \frac{1}{D}\int_b^\xi udr$
$\nabla_h \cdot \int_b^\xi Equation(2)dr$ and using above equation , we get:

$$-\frac{\partial^2 \xi}{\partial t^2} = \nabla_h \cdot D\bar{R} - g\nabla_h \cdot D\nabla_h\xi \tag{7}$$

discretizing the second temporal derivative term, the above equation becomes

$$-\frac{(\frac{\partial \xi}{\partial t})^{n+1} - (\frac{\partial \xi}{\partial t})^n}{\Delta t} = \nabla_h \cdot D\bar{R} - g\nabla_h \cdot D\nabla_h\xi \tag{8}$$

where subscript h means the operator only acts in horizontal direction. The above wave equation can be employed to solve free surface height.

2.2 The New Approach

To solve free surface height, we do finite element discretization to equation(3).

$$\int_V N_i \nabla \cdot u^{n+1} dV + \int_V \nabla N_i \cdot u^{n+1} dV = \int_\Gamma N_i \vec{n} \cdot u^{n+1} d\Gamma \tag{9}$$

where N_i is the finite element test function. As $u \cdot \vec{n} = 0$ at bottom and lateral boundaries and

$$(u \cdot \vec{n})_{surface} = (\frac{\partial \xi}{\partial t})_{surface} \tag{10}$$

by lumping equation(9) in vertical direction and using equation(10), we can get

$$(\int_V (N_i \nabla \cdot u^{n+1} + \nabla N_i \cdot u^{n+1}) dV)_{vertical\ lumped} = (\int_V N_i \frac{1}{D} \frac{\partial \xi}{\partial t} dV)_{vertical\ lumped} \tag{11}$$

During the lumping process, we add the equations up in the rows that correspond to the same vertical column of fluid and assume that the values along each of these columns of fluid is the same. That means, we add together the columns of the matrix that correspond to these.
Assuming ξ^* and u^* are the latest free surface height and velocity respectively, equation(4) becomes

$$\frac{u^* - u^n}{\Delta t} = R - g \nabla \xi^* \tag{12}$$

Subtracting equation(12) from equation(4), we can get

$$\frac{u^{n+1} - u^*}{\Delta t} = -g \nabla \Delta \xi^{n+1} \tag{13}$$

where $\Delta \xi^{n+1} = \xi^{n+1} - \xi^*$
Using equation(8) and equation(11) and approximating $\int_\Gamma \nabla \cdot gD \nabla_h \Delta \xi d\Gamma$ with $\int_V \nabla \cdot gD \nabla \Delta \xi dV$ we get

$$\begin{aligned}(\int_V (\frac{N_i}{D(\Delta t)^2} - \nabla N_i \cdot g \nabla) \Delta \xi^{n+1}) dV)_{vertical\ lumped} = \\ (\int_V \frac{N_i \nabla \cdot u^* + \nabla N_i \cdot u^*}{\Delta t} dV)_{vertical\ lumped} - (\int_V \frac{N_i (\xi^* - \xi^n)}{D(\Delta t)^2} dV)_{vertical\ lumped} \\ + (\int N_i L_s^m \xi^{n+1} dV)_{vertical\ lumped}\end{aligned} \tag{14}$$

The last term on the R.H.S is the added stabilization term. The details about this term please refer to section 6.
On convergence, $\Delta \xi^{n+1} \to 0$, then equation(9) is satisfied for the free surface height

3 A Two-step Approach for the Accurate Balance of Geostrophic Flow

To seek a way to keep the accurate balance of geostrophic flow, the free surface height is decomposed into two parts

$$\xi = \xi_r + \xi_c \tag{15}$$

the gradient of ξ_c will balance Coriolis and buoyancy force term in the momentum equation, e.g.

$$c - g\nabla\xi_c = 0 \tag{16}$$

From the above equation we can see,for linear velocities one would expect to solve for the geo-balance part of free surface height with a quadratic variation. $\int M_i \nabla \cdot (equation(16))dV$ and integrating by parts, we can get

$$\int \nabla M_i \cdot g\nabla\xi_c dV = -\int \nabla M_i \cdot c dV \tag{17}$$

This equation is lumped in vertical and then solved for ξ_c. M_i is a quadratic basis function. However, the rest of the free surface height may then take on any other variation e.g. linear, but there are much better elements than linear elements (P1-P1). Here, a quadratic basis function Q is used for the geo-balance part of free surface height ξ_c.
Assuming the latest velocity and free surface height are u^* and $\xi^* = \xi_r^* + \xi_c$ respectively, from equation(2) we can get

$$\frac{u^* - u^n}{\Delta t} = R - g\nabla(\xi_r^* + \xi_c) \tag{18}$$

$$\frac{u^{n+1} - u^n}{\Delta t} = R - g\nabla_h(\xi_r^{n+1} + \xi_c) \tag{19}$$

by subtracting equation(18) from equation(19), we get

$$\frac{u^{n+1} - u^*}{\Delta t} = -g\nabla(\xi_r^{n+1} - \xi_r^*) \tag{20}$$

Similar to the derivation of equation(14) and noticing that $\xi = \xi_c + \xi_r$, we get

$$\begin{aligned}(\int_V (\frac{N_i}{D(\Delta t)^2} - \nabla N_i \cdot g\nabla + L_s)\Delta\xi_r^{n+1} dV)_{vertical\ lumped} = \\ (\int_V \frac{N_i \nabla \cdot u^* + \nabla N_i \cdot u^*}{\Delta t} dV)_{vertical\ lumped} - (\int_V \frac{N_i(\xi_r^* - \xi^n)}{D(\Delta t)^2} dV)_{vertical\ lumped} \\ + (\int_V N_i L_s^m \xi_r^{n+1} dV)_{vertical\ lumped}\end{aligned} \tag{21}$$

where $\Delta\xi_r^{n+1} = \xi_r^{n+1} - \xi_r^*$. The last term on the R.H.S is the added stabilization term. The details about this term please refer to section 6.
Firstly, we solve equation(16) for ξ_c, then solve equation(18) for u^*. After that,we solve equation(21) for $\Delta\xi_r^{n+1}$. If $\Delta\xi_r^{n+1}$ approaches zero, we have already got the solution: $\xi^{n+1} = \xi_r^* + \xi_c^{n+1}$ and $u^{n+1} = u^*$. Otherwise, we need to update c, r and ξ_r^* and repeat above iteration.

4 Implicit Free Surface Treatment of Coriolis and Frictional/Absorption Terms

To do the implicitness of Coriolis term and absorption term, The momentum equation (2) is rewrittem in the following form:

$$\frac{u^{n+1} - u^n}{\Delta t} = r - 2\theta_c(\Omega \times u^{n+1} - \Omega \times u^n) - 2\Omega \times u^n - \sigma\theta_a(u^{n+1} - u^n) - \sigma u^n - g\nabla\xi^{n+1} \tag{22}$$

That is

$$C\frac{u^{n+1}-u^n}{\Delta t} = R' - g\nabla\xi^{n+1} \tag{23}$$

where $R' = r - 2\Omega \times u^n - \sigma u^n$ and $C = (I + 2\theta_c \Delta t \Omega \times I + \theta_a \sigma I)$. I is the identity matrix. θ_c and θ_a are the implicit coefficient for Coriolis term and absorption term respectively. $0 \le \theta_c \le 1$ and $0 \le \theta_a \le 1$. $\theta_c = 0$ represents explicit Coriolis term and $\theta_c = 1$ means fully implicit Coriolis term. $\theta_a = 0$ represents explicit absorption term and $\theta_a = 1$ means fully implicit absorption term.
Assuming the latest values of u^{n+1} and ξ^{n+1} are u^* and ξ^* respectively, then

$$C\frac{u^*-u^n}{\Delta t} = R' - g\nabla\xi^* \tag{24}$$

and

$$\frac{\xi^{n+1}-\xi^n}{D\Delta t} + \nabla\cdot u^{n+1} = 0 \tag{25}$$

Subtract equation (24) from equation (23),

$$C\frac{u^{n+1}-u^*}{\Delta t} = -g\nabla_h(\xi^{n+1}-\xi^*) \tag{26}$$

Using the same method for the derivation of equation(14), we get

$$\begin{aligned}
(\int (\frac{N_i}{D(\Delta t)^2} - \nabla N_i \cdot gC^{-1}\nabla + L_s)\Delta\xi^{n+1}dV)_{vertical\ lumped} = \\
(\int \frac{N_i\nabla\cdot u^* + \nabla N_i\cdot u^*}{\Delta t}dV)_{vertical\ lumped} - (\int \frac{N_i(\xi^*-\xi^n)}{D(\Delta t)^2}dV)_{vertical\ lumped} \\
+ (\int N_i L_s^m \xi^{n+1}dV)_{vertical\ lumped}
\end{aligned} \tag{27}$$

where $\Delta\xi^{n+1} = \xi^{n+1} - \xi^*$. The last term on the R.H.S is the added stabilization term. The details about this term please refer to section 6.
We can take ξ^n as the initial value of ξ^* and get u^* by using equation (24). The updated free surface height can be got by solving equation(27). If $\Delta\xi^{n+1}$ approaches zero, we have got the solution $\xi^{n+1} = \xi^*$ and $u^{n+1} = u^*$. Otherwise, the above process need to be repeated with $\xi^* = \Delta\xi^{n+1} + \xi^*$ and $u^* = u^{n+1}$.

5 Wetting and Drying

To deal with wetting and drying processes, we try to keep a thin layer of of water on the dried element. This will make the bulk of the code can deal with the wet and dry area in the same way. It will be easy for people to extend all the techniques designed for wet area to dry area. Here two kinds of method will be used individually or together.

5.1 Artificial Absorbing Term Friction

When a element is going to dry out, the water depth will decrease to zero. To keep a thin layer of water, An extra absorption term will be implicitly added into the momentum equation. This term has

the property of damping the velocity if water layer decrease to a certain depth, say d_0, but will rapidly decay to zero when water depth become greater than d_0.

$$\sigma = K\frac{\sqrt{E_0 E_1 g d}}{\Delta x} \tag{28}$$

where σ is the artificial absorption term. Realizing add σu into the L.H.S of the momentum equations, Δx is the maximum distance from the current node to the local surrounding nodes. d is the water depth at current node point. d_0 is the predefined critical water depth (default value is 0.1). E_0 is a constant (default value is 10.). g is the gravity acceleration. k is another constant with 1.0 as its default value. $E_1 = e^{-E_0(\frac{d-d_0}{d_0})}$

5.2 Method of Artificial Pressure Term

Have the same objectives of keeping a thin layer of water in dried area, a extra pressure term can be added to suck water back to the area where is tending to dry out. This pressure term must decay to zero in wet area.

$$P_{wd} = -g d_0 \exp^{-\frac{E_0}{d_0}(d-d_0)} \tag{29}$$

P_{wd} is the Artificial Sucking Pressure term. All the other variables and constants has the same definitions and values as those in equation(28).

6 Sub-grid Stabilization for Free Surface Solver

In the process of solving equation(14) together with equation(4), some spurious noise might be generated through the discretization scheme. A stabilization term can be added to eliminate this problem. This term can be of 2^{nd}, 4^{th} or higher order. This stabilization method should not hurt the property of consistency.

Add the stabilization term into the free surface height equations equation(14,21,27)

$$\frac{\frac{\xi^{n+1}-\xi^{n}}{D\Delta t} - \nabla \cdot u^n}{\Delta t} = \nabla \cdot R - \nabla \cdot g\nabla\xi + L_s^m \xi \tag{30}$$

where $L_s^m \xi$ is the sub-grid stabilization term. m is the order of it.

In the following part of this section, we will show how the 4^{th} order stabilization is added.

$$L_s^4 \xi = (\nabla \cdot C_1 \nabla \xi)_1 - (\nabla \cdot C_1 \nabla \xi)_2 \tag{31}$$

The finite element discretization of the stabilization term in the above equation are

$$\int N_i (\nabla \cdot C_1 \nabla \xi)_1 dV = -\sum_j \int \nabla N_i \cdot C_1 \nabla N_j \xi_j dV + b.c's \tag{32}$$

in above equation, the 2^{nd} order term has been integrated by parts.

$$\int N_i (\nabla \cdot C_1 \nabla \xi)_2 dV = \int a \cdot \nabla N_i dV + b.c's \tag{33}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{34}$$

$$a_q = M^{-1} B_{ij}^q \xi \tag{35}$$

$$B_{ij}^q = \int N_i \frac{\partial N_j}{\partial x_q} dV \tag{36}$$

$$M = \int N_i (1/C_1) N_j dV \tag{37}$$

$C_1 = \frac{\Delta x \sqrt{gD}}{D \Delta t}$, D is the water depth, Δx is the horizontal length scale of local element size.
It can be proved that equation(32) will construct a 4^{th} order stabilization term. The two terms here have the same analytic expression, so the stabilization term won't hurt the property of consistency of the free surface height equation.
The 2^{nd} and higher order stabilization can be derived in the similar way.